Poncelet Families and the Triangle of Triangles

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Abstract

Poncelet's Theorem states that if a polygon is inscribed in one conic and circumscribes another conic, then the polygon is part of an infinite family of polygons that inscribe and circumscribe the same two conics [1]. We define a *Poncelet family* to be a family of triangles that inscribe and circumscribe the same two circles. We plot these families on a "moduli space" of triangles called the Triangle of Triangles, and show that they are the level curves of a certain surface. We then consider them more broadly over the Riemann sphere, to introduce some new families and unify the presentation. Finally, we explicitly parameterize these families on the Triangle of Triangles using a parameter that is natural to the Poncelet setup.

Parameter Space of Triangles

We define a parameter space of triangles to be the triples of angles that sum to π, given in $\mathbb{R}^3$ by the plane $x + y + z = \pi$. If we restrict the plane to the first octant, it forms a triangle, known as the *Triangle of Triangles*, shown in Figure 1.

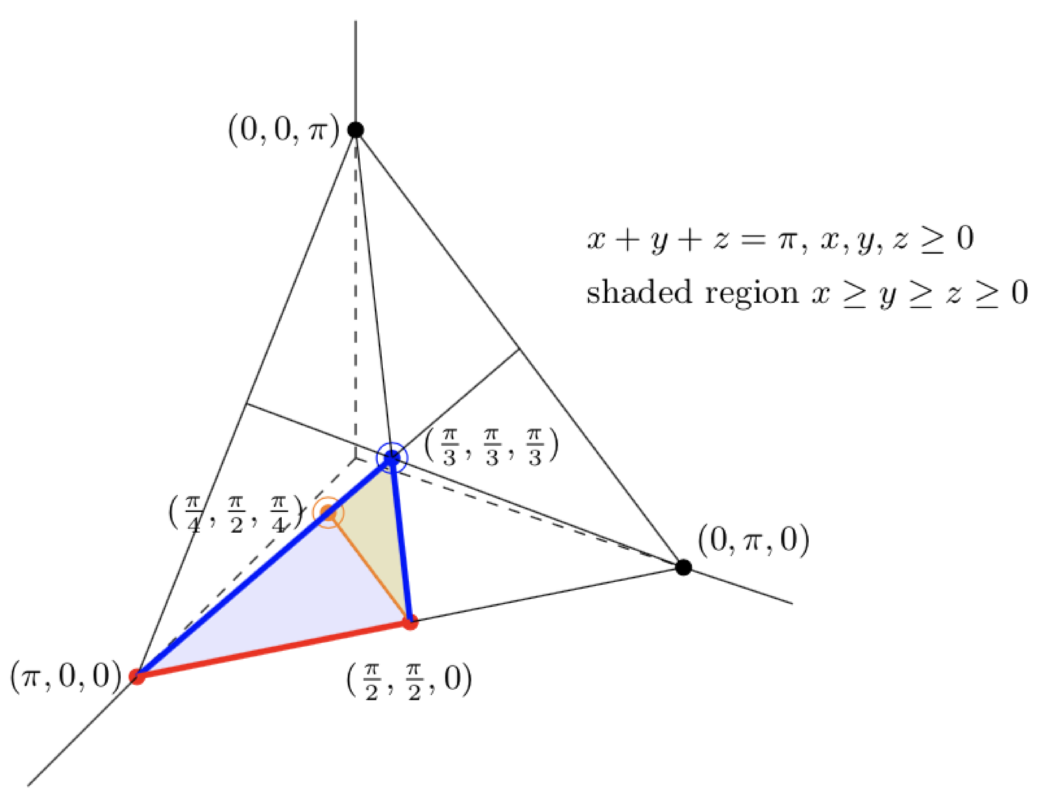

Figure 1: The Triangle of Triangles

Triangles on the Riemann Sphere

We consider the labeled, oriented $\triangle ABC$ by its three positive interior angles, (α, β, γ). Observe that $\triangle ABC$ inscribes the *outcircle* centered at O and circumscribes the *incircle* centered at I_0, as shown in Figure 2. Let the *quasi-triangle* be three lines that lie tangent to the *excircle* centered at I_A, and let it be given by $(-\alpha, \pi - \beta, \pi - \gamma)$.

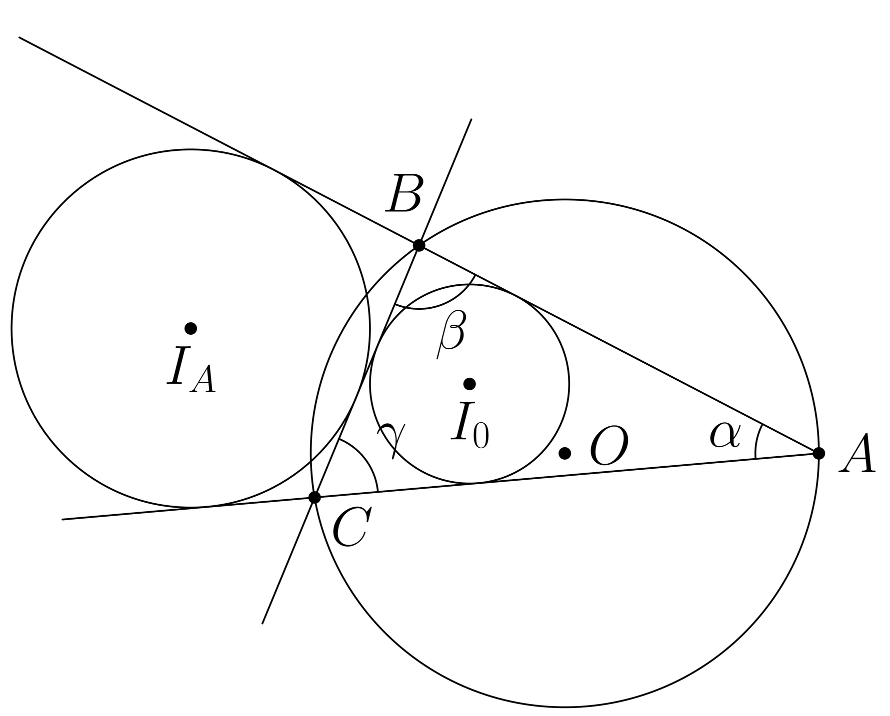

Figure 2: $\triangle ABC$ and a corresponding quasi-triangle

Poncelet's Theorem

Poncelet's Theorem states that *if a polygon is inscribed in one conic section and circumscribes another conic section, then the polygon is a part of an infinite family of polygons that all inscribe and circumscribe the same two conics.*

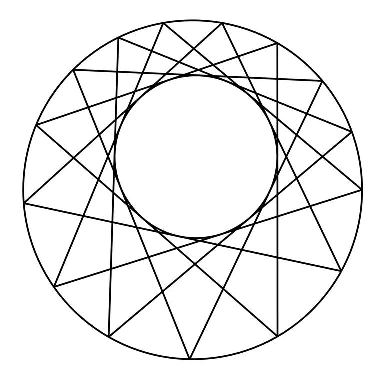

Figure 3: A Poncelet Family of Triangles

Define a *Poncelet family* to be a family of triangles (or quasi-triangles) that inscribe and circumscribe the same two circles, as shown in Figure 3.

Poncelet Families as Level Curves

We would like to plot Poncelet families as smooth curves in our parameter space of triangles. Expanding the parameter space to the entire plane $x + y + z = \pi$ in $\mathbb{R}^3$, we find a surface over the plane whose level curves are Poncelet families. Any triangle $\triangle ABC$ circumscribes a unique incircle and inscribes a unique circumcircle. The radii of these two circles are related by the formula

$$\frac{r}{R} = 4 \sin\frac{\alpha}{2} \sin\frac{\beta}{2} \sin\frac{\gamma}{2}, \quad (1)$$

where α, β, and γ are the interior angles of the triangle, r is the inradius, and R is the outradius. It follows that level curves of the surface given by $z = f(x, y) = \frac{r}{R}$ are Poncelet families.

Poncelet Surfaces

Figure 4 plots the surface whose level curves are Poncelet families. If $\frac{r}{R} > 0$, then the Poncelet family is given by an incircle. If $\frac{r}{R} < 0$, then the Poncelet family is given by an excircle, and the members of the family are quasi-triangles.

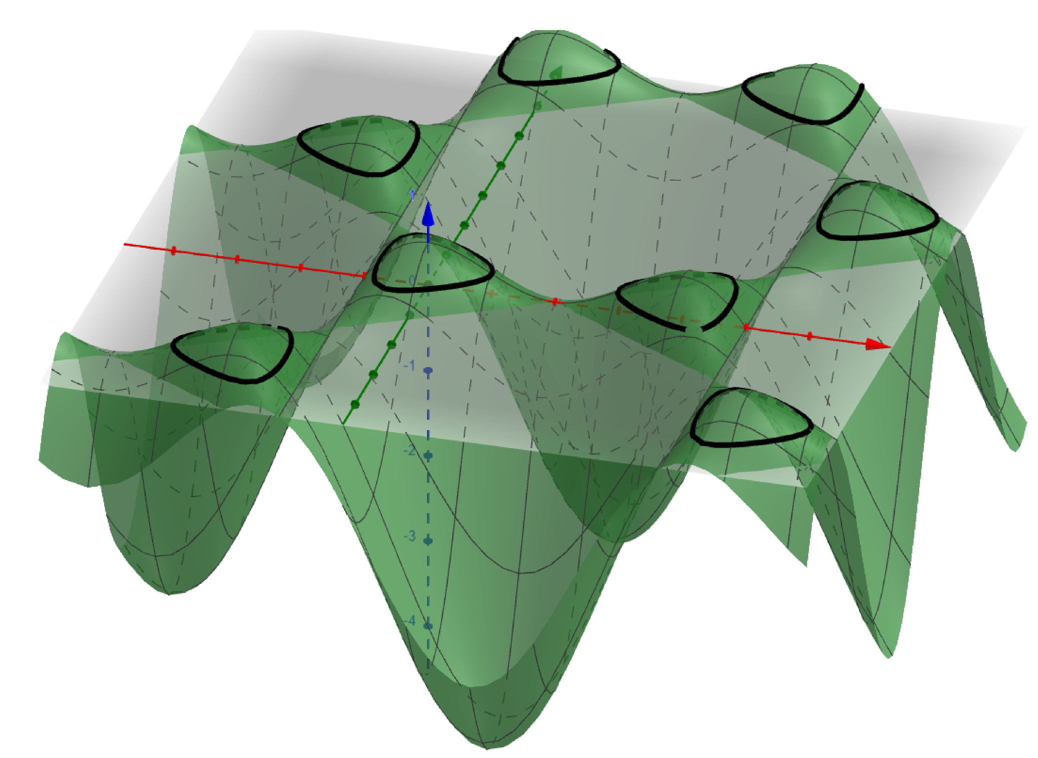

Figure 4: Plot of Incircle Poncelet surface, $z = \frac{r}{R}$

As the incircle or excircle radius of a Poncelet family approaches zero, the level curve *degenerates* to a triangle or hexagon, whose outlines can be seen in Figure 4.

Intersecting Poncelet Families

Every triangle has one incircle and three excircles, hence every triangle is a member of exactly four Poncelet families, one triangular (as in Figure 4) and three hexagonal. The surfaces representing these four different classes of Poncelet families are four translated copies of the $z = \frac{r}{R}$ surface.

Poncelet Families on the Riemann Sphere

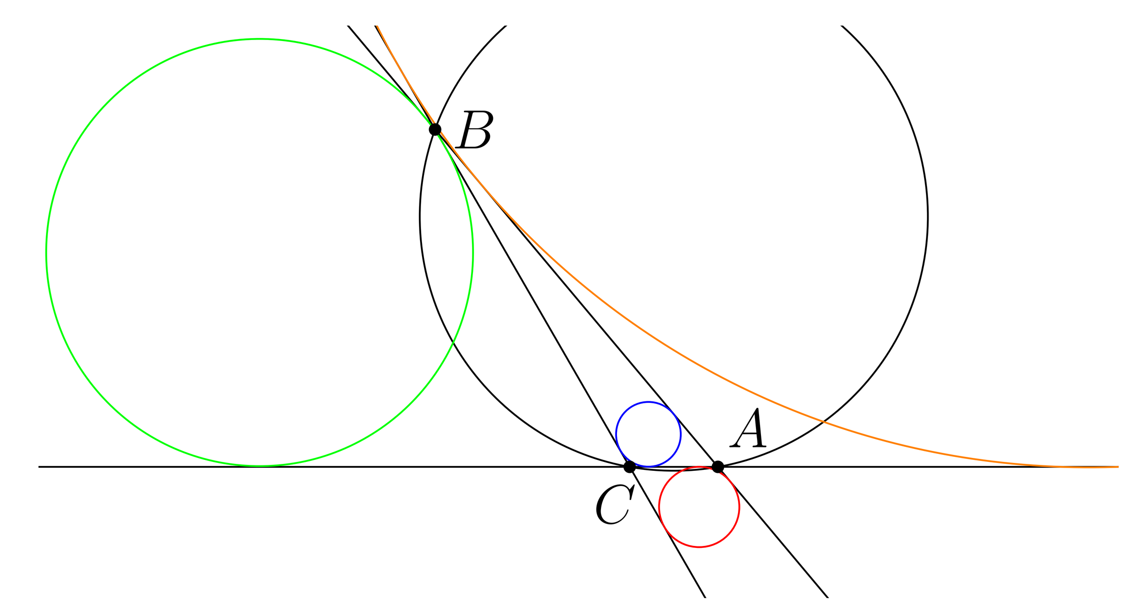

Figure 5: A Poncelet configuration in the plane.

The three quasi-triangles in Figure 5 become triangles when infinity is adjoined to the plane, to form the *Riemann sphere* (see Figure 6).

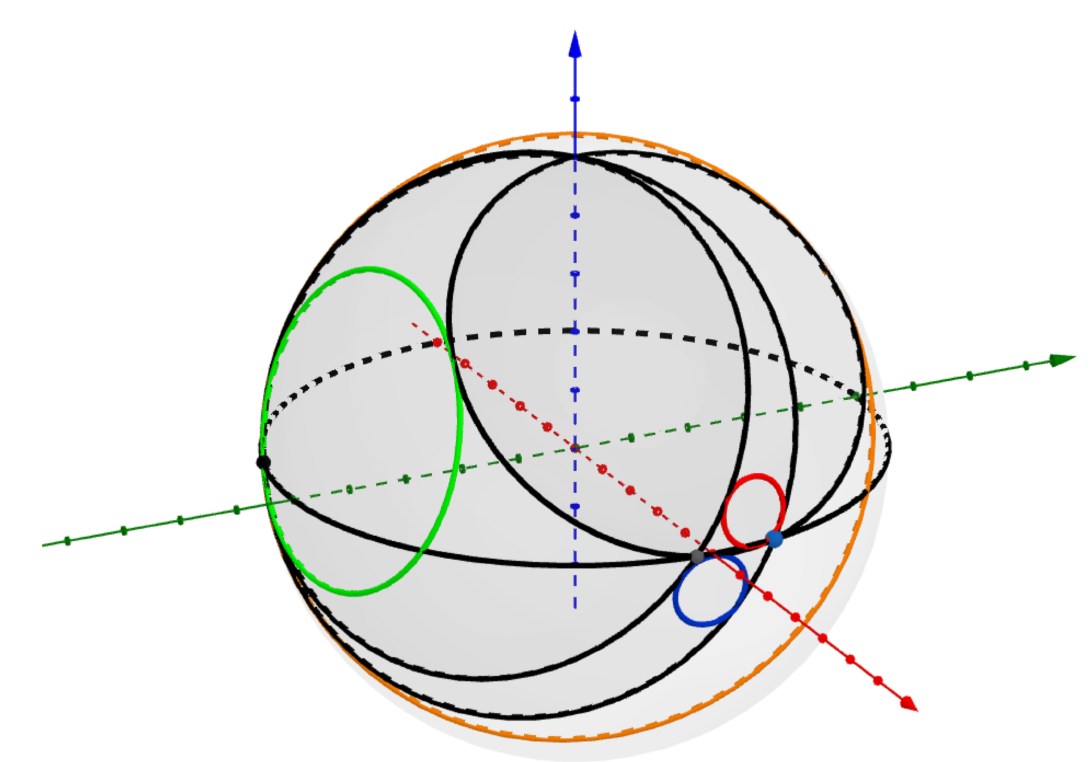

Figure 6: A Poncelet configuration on the Riemann sphere.

References

[1] King, J.L. (1994). Three Problems in Search of a Measure. *The American Mathematical Monthly*, **101**(7), 609-628, doi: 10.1080/00029890.1994.11997003.

Acknowledgments

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