Three Moduli Spaces of Triangles

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Abstract

In his Pillow Problems [2], Lewis Carroll asked, "what is the probability that a random triangle is obtuse?" To answer this question requires defining a "random" triangle, which many authors have explored [3, 4, 5]. We consider labeled, oriented triangles and examine the geometry of their direct similarity classes under three different classical similarity theorems: side-side-side, side-angle-side, and angle-angle-angle. Each theorem results in a topologically distinct moduli space: a 2-sphere, Klein bottle, and Clifford torus of triangles.

Key Definitions

(a) A **triangle** is a point $(A, B, C) \in \mathbb{C}^3$, where each coordinate is a vertex of the triangle. It has opposite side lengths a, b, c, and corresponding interior angles α, β, γ.

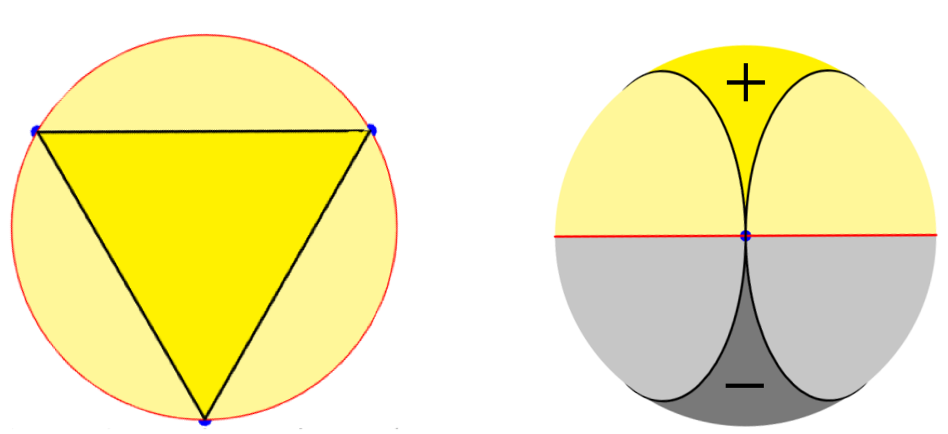
(b) A triangle is **(non)degenerate** if its area is (non)zero.

(c) See Figure 2. A degenerate triangle $(A, B, C) \in \mathbb{C}^3$ is of
  (i) **mult. 1** if A, B, C are distinct collinear points.
  (ii) **mult. 2** if exactly two of the vertices are equal.
  (iii) **mult. 3** or **trivial** if all three vertices are equal.

(d) A nondegenerate triangle $(A, B, C) \in \mathbb{C}^3$ is **positively (negatively) oriented** if the curve $A \to B \to C \to A$ of line segments in the complex plane is positively (negatively) oriented.

Construction of Moduli Spaces

SSS Construction — 2-sphere



(a) Top View (b) Side View

Figure 1: The Sphere of Triangles. The positively oriented, negatively oriented, acute, and obtuse triangles appear in the yellow, gray, dark, and light regions, respectively. The right, mult. 1 degenerate, and mult. 2 degenerate triangles appear in black, red, and blue, respectively.

Let $p \in \mathbb{C}^3$ be a triangle. We prove that the *(labeled, oriented, direct) similarity class* of p is

$$[p] = \{zp + t(1,1,1) : z \in \mathbb{C}^\times, t \in \mathbb{C}\} \subset \mathbb{C}^3.$$

The dual of $[p]$ is a (complex) line in $\mathbb{C}^{3*}$ that lies on a plane. Therefore, the similarity classes are parameterized by the Grassmannian $\mathrm{Gr}(1, \mathbb{C}^2)$. We map to S^2, the 2-sphere, using the Hopf map

$$\mathrm{Gr}(1, \mathbb{C}^2) \to S^2$$
$$[a] \mapsto a^{-1}\mathbf{i}a.$$

See [3] and [4].

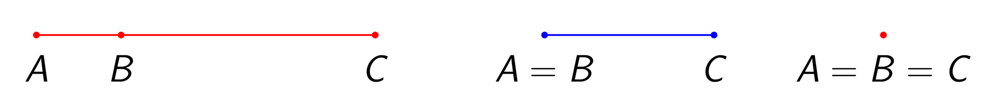


Figure 2: Example multiplicity 1, 2, and 3 degenerate triangles

SAS Construction — Klein bottle

We begin with a triple of real numbers, (θ, a, b). a and b represent side lengths (possibly negative, but not both zero) and θ represents their enclosed angle (mod 2π). To mod out by scaling, we projectivize the second two coordinates:

$$(\theta, a, b) \sim (\theta, \lambda a, \lambda b) \text{ for all } \lambda \in \mathbb{R}/\{0\}.$$

Then we account for the relation (Figure 3):

$$(\theta, a, b) \sim (\theta + \pi, -a, b).$$

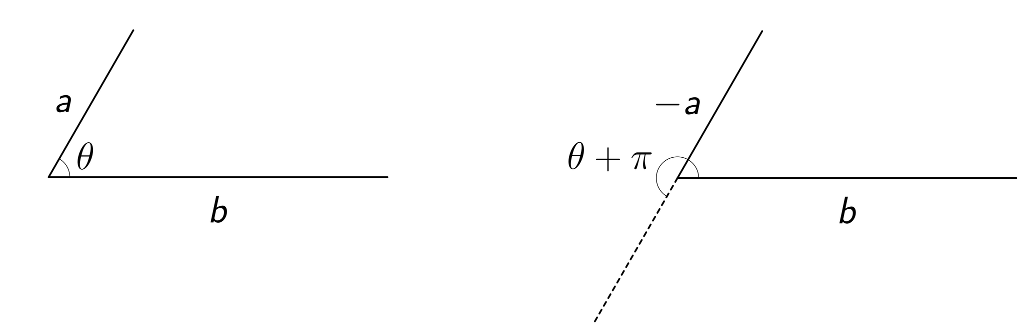


Figure 3: SAS Relation

After parameterizing the square $[0, \pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ via the map $(\theta, a, b) \mapsto (\theta, \tan^{-1}(\frac{b}{a}))$ and gluing appropriately, we discover a Klein bottle 🤯:

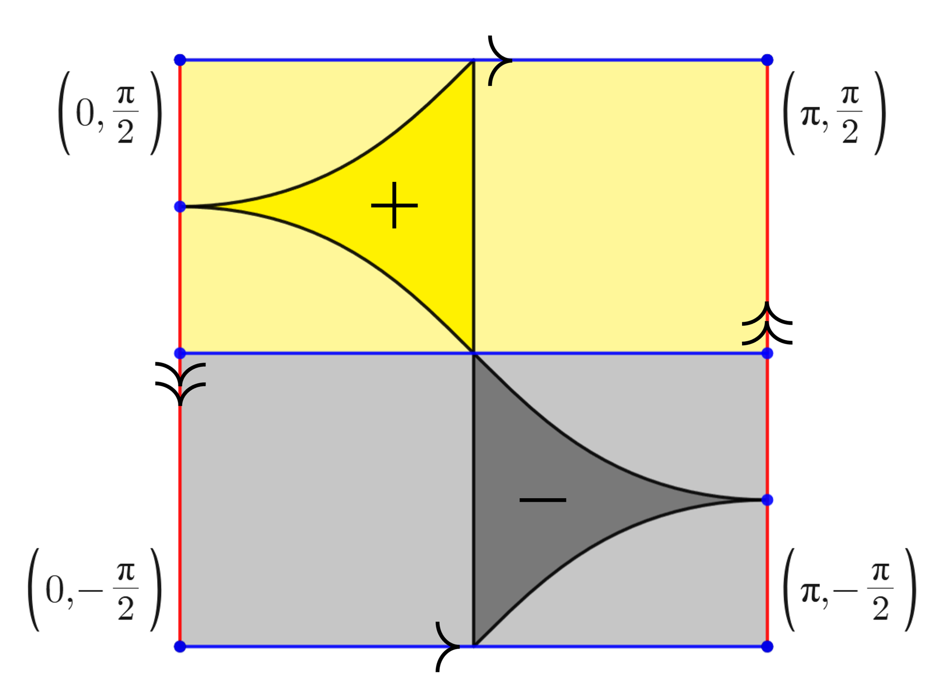


Figure 4: The Klein bottle.

AAA Construction — Torus

Two triangles are similar if their interior angles are equal, in the same order. It is shown in [1] that the planes $\alpha + \beta + \gamma = \pm\pi$ glue together to form a torus:

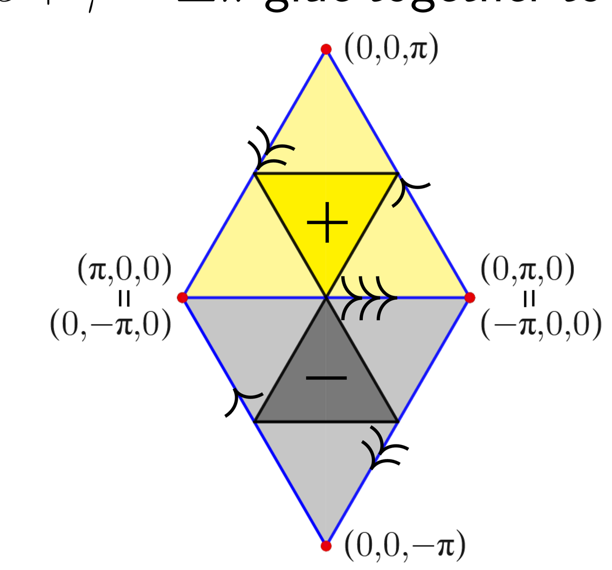


Figure 5: The Torus of Triangles [1, Figure 4.2]

Observations

Metrics

(a) On the sphere of triangles, we use the natural metric of geodesics, which is induced by the Fubini-Study metric on $\mathrm{Gr}(1, \mathbb{C}^2)$ and preserved by the Hopf map.

(b) For SAS similarity, the Klein bottle admits the flat or Euclidean metric on Figure 4.

(c) For AAA similarity, the torus admits the flat or Euclidean metric on Figure 5 [1].

Probabilities

The metrics defined above allow us to compute probabilities. Let $P(\mathrm{O})$ be the probability that a triangle is obtuse.

(a) SSS: $P(\mathrm{O}) = \frac{3}{4}$.

(b) SAS: $P(\mathrm{O}) \approx 0.84$.

(c) AAA: $P(\mathrm{O}) = \frac{3}{4}$ (see [1].)

Degenerate Triangles

Degenerate triangles appear differently under the three constructions. Multiplicity 1/3 and 2 degenerate triangles are shown in red and blue in Figures 1, 4, and 5.

Theorem	Mult. 1	Mult. 2	Mult. 3
SSS	3 curves	3 points	N/A
SAS	3 curves	2 curves, 1 point	N/A
AAA	N/A	3 curves	1 point

Table 1: Characterization of degenerate triangles.

Comparison of Spaces

(a) The sphere of triangles admits a transitive continuous group action by $SO(3)$, but is not a PHS.

(b) We do not have a natural transitive group action on the Klein bottle. Investigation ongoing.

(c) The torus of triangles is a PHS for the Clifford torus, a compact abelian Lie group [1].

Further Work

(a) We would like to induce the three similarity theorems on the same original space of triangles.

(b) We would like to find a transitive group action on the Klein bottle.

References

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