Linear Optimization Degrees of Varieties

Madeleine Goertz

UC Berkeley

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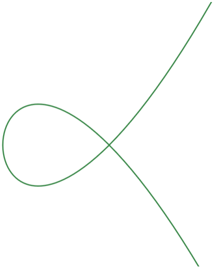
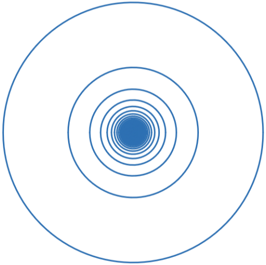
What is a variety?

A subset $X \subset \mathbb{C}^n$ is an (affine) variety if X is the zero set of an ideal of polynomials $I \subset \mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$.



$\left\{x_2^2 - x_1^3 - x_1^2 = 0\right\}$
A variety!



$\bigcup_{r=1}^{\infty} \left\{(x_1^2 + x_2^2 - 1/r^2) = 0\right\}.$
Not a variety!

Optimization on Varieties

For a fixed point $u \in \mathbb{C}^n$ and $f_u(x) \in \mathbb{C}[u, x]$, the optimal value of f on a variety $X \subset \mathbb{C}^n$ is

$$\min \ f_u(x) \quad \text{subject to} \quad x \in X.$$

Here u has real coordinates, $f_u(x)$ has real coefficients, and X is defined over $\mathbb{R}$.
E.g.: Euclidean distance to u: $f_u(x) = \sum_{i=1}^n (x_i - u_i)^2$.

How to solve:

1. Compute the **critical points** using Lagrange multipliers.
2. Then compute minimum over finitely many critical points.

Often, for generic u, the number of critical points doesn't depend on u. Call this the algebraic degree of the optimization problem.

Linear Optimization on Varieties

(Sectional) LO Degrees

Fix $X \subset \mathbb{C}^n$ an irreducible affine variety with $\dim(X) = d$.

The linear optimization (LO) degree of X, $\mathrm{LOdeg}(X)$, is the number of critical points of a generic linear functional on $\mathbb{C}^n$ restricted to X_{sm}.

For $i = 0, \ldots, d$, the i-th sectional LO degree of X is

$$\sigma_i(X) := \mathrm{LOdeg}(X \cap H_1 \cap \cdots \cap H_i),$$

where $H_1, \ldots, H_i$ are generic affine hyperplanes in $\mathbb{C}^n$.

$\sigma_0(X) = \mathrm{LOdeg}(X)$, $\sigma_d(X) = \deg(X)$.

Application: Polyhedral Norm Optimization

Let $u \in \mathbb{C}^n$ be a point. Find the closest point on X to u in a polyhedral norm, a norm whose unit ball in the Euclidean metric is a polytope P.

One algorithm involves solving a linear optimization problem for each of the faces of P.

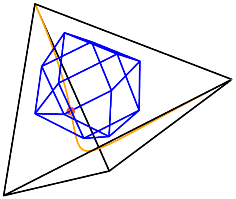


Figure: Wasserstein ball for discrete metric touching the twisted cubic.[1]

This has applications to machine learning.

[1] G DePaul, S Hoșten, N Metya, and I Nometa. Degrees of the Wasserstein distance to small toric models, Algebraic Statistics 15 no. 2 (2024) 249–267.

Application: Polyhedral Norm Optimization

The algebraic degree of each facial linear optimization problem is bounded by a polar degree of X.[2]

For X an independence model, this bound is experimentally not tight.

Sectional LO degrees improve these bounds.

[2] T Ö Çelik, A Jamneshan, G Montúfar, B Sturmfels, and L Venturello. Wasserstein distance to independence models, Journal of Symbolic Computation 104 (2021) 855–873.

Technical Tools

Projective Space

Projective space $\mathbb{P}^n$ captures the "points at infinity" in $\mathbb{C}^n$.

$\mathbb{P}^n$ is the space of lines through the origin in $\mathbb{C}^{n+1}$. Use homogeneous coordinates:

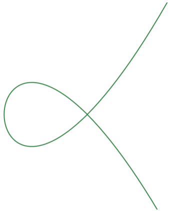
$$[x_0 : x_1 : \cdots : x_n] \sim [\lambda x_0 : \lambda x_1 : \cdots : \lambda x_n] \quad \text{for all } \lambda \in \mathbb{C} \setminus \{0\}\,.$$

$\mathbb{P}^n$ is covered by copies of $\mathbb{C}^n$, called affine charts.

Two parallel lines in $\mathbb{C}^n$ intersect in $\mathbb{P}^n$ at a point at infinity.

Projective Varieties

A subset $Y \subset \mathbb{P}^n$ is a (projective) variety if Y is the zero set of an ideal $I \subset \mathbb{C}[x_0, \ldots, x_n]$ of homogeneous polynomials: every term in the polynomial has the same degree.



$x_0 x_2^2 - x_1^3 - x_0 x_1^2 = 0$, on the affine chart $x_0 = 1$.

Projective Closure

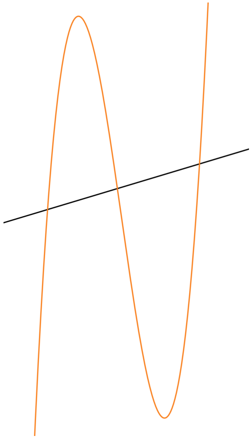
Given an affine variety $X \subset \mathbb{C}^n$, the projective closure $\overline{X}$ of X is the smallest projective variety $\overline{X} \subset \mathbb{P}^n$ containing X.

Obtain the equations of $\overline{X}$ by homogenizing the equations of X:

$$x_2^2 - x_1^3 - x_1^2 \quad \rightsquigarrow \quad x_0 x_2^2 - x_1^3 - x_0 x_1^2$$

The Degree of a Variety

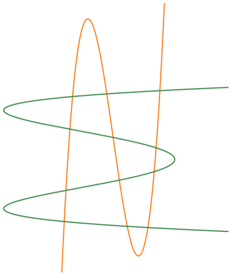
The degree of a variety X of dimension d is the number of points at which a generic linear space of complementary dimension intersects X.



A line intersects a cubic at 3 points.

Bezóut's Theorem

In $\mathbb{P}^n$, n generic hypersurfaces $X_1, \ldots, X_n$ intersect at $\deg(X_1) \cdots \deg(X_n)$ points.



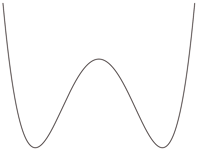
A cubic and a quartic in the plane intersect at $3 \cdot 4$ points.

Dual Space

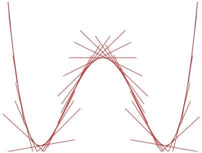
The dual space $(\mathbb{P}^n)^\vee$ is the space of hyperplanes in $\mathbb{P}^n$:

$$\{a_0x_0 + \cdots + a_nx_n = 0\} \subset \mathbb{P}^n \quad \leftrightarrow \quad [a_0 : \cdots : a_n] \in (\mathbb{P}^n)^\vee.$$

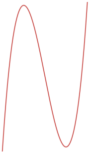
The dual variety of a projective variety $Y \subset \mathbb{P}^n$ is the variety $Y^\vee \subset (\mathbb{P}^n)^\vee$ of all hyperplanes in $\mathbb{P}^n$ tangent to Y.



The variety Y



Tangent lines to Y



The dual variety $Y^\vee$

Discriminants

When the dual variety $Y^\vee \subset (\mathbb{P}^n)^\vee$ is a hypersurface, its defining polynomial $\Delta_Y \in \mathbb{C}[a_0, \ldots, a_n]$ is the discriminant of Y.

The variety $Y \subset \mathbb{P}^2$ given by $x_0x_2 - x_1^2 = 0$ is parametrized by $[x_0 : x_1 : x_2] = [1 : t : t^2]$.

$a = [a_0 : a_1 : a_2] \in (\mathbb{P}^2)^\vee$ corresponds to the hyperplane $H_a = \{a_0x_0 + a_1x_1 + a_2x_2 = 0\} \subset \mathbb{P}^2$.

1. H_a intersects Y iff $f_a(t) = a_0 + a_1t + a_2t^2 = 0$.
2. H_a is tangent to Y iff $f_a(t)$ has a double root.

So $\Delta_Y(a_0, a_1, a_2) = a_1^2 - 4a_0a_2$. This is "$b^2 - 4ac$".

The Conormal Variety

$Y \subset \mathbb{P}^n$ irreducible projective variety. The conormal variety of Y:

$$N_Y = \overline{\{(x, H) \in \mathbb{P}^n \times (\mathbb{P}^n)^\vee \mid x \in Y_{sm} \text{ and } H \text{ is tangent to } Y \text{ at } x.\}}$$

$$\mathbb{P}^n \supset Y \xleftarrow{\text{pr}_1} N_Y \xrightarrow{\text{pr}_2} Y^\vee \subset (\mathbb{P}^n)^\vee$$

N_Y is irreducible of dimension $n-1$.

Intersection theory

X a smooth variety. The Chow ring $A(X)$ of X is a graded ring which lets us study how subvarieties of X intersect up to a notion of equivalence.

$A(\mathbb{P}^n) \cong \mathbb{Z}[\zeta]/\left\langle \zeta^{n+1} \right\rangle$, $\zeta = [\mathbb{P}^{n-1}]$ is the class of a hyperplane.

For a projective variety $X \subset \mathbb{P}^n$ of $\dim(X) = d$, the class of X in $A(\mathbb{P}^n)$ is

$$[X] = \deg(X) \cdot \zeta^{n-d} \in A_d(\mathbb{P}^n).$$

The Chow ring of $\mathbb{P}^n \times (\mathbb{P}^n)^\vee$ is

$$A(\mathbb{P}^n \times (\mathbb{P}^n)^\vee) \cong \mathbb{Z}[\zeta, \xi]/\left\langle \zeta^{n+1}, \xi^{n+1} \right\rangle,$$

ζ and ξ are the pullbacks of hyperplane classes in $\mathbb{P}^n$ and $(\mathbb{P}^n)^\vee$.

Polar Degrees

$Y \subset \mathbb{P}^n$ irreducible projective variety. The polar degrees of Y, $\delta_i(Y)$, defined as bidegrees of the conormal variety $N_Y \subset \mathbb{P}^n \times (\mathbb{P}^n)^\vee$:

$$[N_Y] = \sum_{i=0}^{\dim(Y)} \delta_i(Y)[\mathbb{P}^i \times \mathbb{P}^{n-1-i}] \in A_{n-1}(\mathbb{P}^n \times (\mathbb{P}^n)^\vee).$$

$L_{n-i} \subset \mathbb{P}^n$, $L_{i+1} \subset (\mathbb{P}^n)^\vee$ generic linear subspaces of dimension $n-i$ and $i+1$.
$L_{n-i} \times L_{i+1}$ intersects N_Y transversally at $\delta_i(Y)$ points.

$\delta_{\dim(Y)}(Y) = \deg(Y)$, $\delta_e(Y) = \deg(Y^\vee)$, $\delta_i(Y) = 0\ \forall i < e$, $e = \operatorname{codim}(Y^\vee) - 1$.

LO degree & Discriminants

Coordinates on $\mathbb{C}^n$: $x_1, \ldots, x_n$. $\overline{X} \subset \mathbb{P}^n$ the projective closure of X, obtained by homogenizing equations of X by x_0. Dual coordinates $a_0, \ldots, a_n$ on $(\mathbb{P}^n)^\vee$.

Theorem 1 (G. '26+)

Suppose $\overline{X}^\vee$ is a hypersurface with defining equation $\Delta_{\overline{X}}$, a homogeneous irreducible polynomial in the variables $a_0, \ldots, a_n$ of total degree $\delta_0(\overline{X})$. Then the following are equal:

1. *The linear optimization degree* $\mathrm{LOdeg}(X)$.
2. *The degree of $\Delta_{\overline{X}}$ in the variable a_0, denoted* $\deg_{a_0}(\Delta_{\overline{X}})$.

Rostalski & Sturmfels assume X defined over $\mathbb{R}$ and $X(\mathbb{R}) \subset \mathbb{R}^n$ irreducible, smooth, compact affine variety.[3]

[3] P Rostalski and B Sturmfels. Dualities in convex algebraic geometry, Rendiconti di Matematica e delle sue Applicazioni. Serie VII 30 nos. 3–4 (2010) 285–327.

Example: (2_3)

Let $X \subset \mathbb{C}^4$ be the 1-dimensional variety with ideal

$$I(X) = \left\langle 3x_1x_3 - x_2^2, 9x_1x_4 - x_2x_3, 3x_2x_4 - x_3^2, x_1 + x_2 + x_3 + x_4 - 1 \right\rangle .$$

Then $\overline{X} \subset \mathbb{P}^4$ has ideal

$$I(\overline{X}) = \left\langle 3x_1x_3 - x_2^2, 9x_1x_4 - x_2x_3, 3x_2x_4 - x_3^2, x_1 + x_2 + x_3 + x_4 - x_0 \right\rangle .$$

Example: (2_3)

$\Delta_{\overline{X}}$ is a polynomial in $a_0, \ldots, a_4$ with 27 terms of total degree 4 and its degree in a_0 is 2:

$$\begin{aligned}
&(a_1^2 - 6a_1a_2 + 9a_2^2 + 6a_1a_3 - 18a_2a_3 \\
&\qquad +9a_3^2 - 2a_1a_4 + 6a_2a_4 - 6a_3a_4 + a_4^2) \cdot a_0^2 \\
&+ (4a_2^3 - 6a_1a_2a_3 - 6a_2^2a_3 + 12a_1a_3^2 - 6a_2a_3^2 + 4a_3^3 + 2a_1^2a_4 \\
&\quad - 6a_1a_2a_4 + 12a_2^2a_4 - 6a_1a_3a_4 - 6a_2a_3a_4 + 2a_1a_4^2) \cdot a_0 \\
&\qquad - 3a_2^2a_3^2 + 4a_1a_3^3 + 4a_2^3a_4 - 6a_1a_2a_3a_4 + a_1^2a_4^2
\end{aligned}$$

The LO degree of X is $\sigma_0(X) = 2$.

$\delta_0(\overline{X}) = 4$. So the gap between δ_0 and σ_0 is 4.

The LO gaps

$H_\infty = \{x_0 = 0\} \in (\mathbb{P}^n)^\vee$ is the hyperplane at infinity.

Set $B = \mathbb{P}^n \times \{H_\infty\}$.

$F = \mathrm{pr}_2^{-1}(H_\infty) \subset B \subset N_{\overline{X}}$ is the fiber of $\mathrm{pr}_2 \colon N_{\overline{X}} \to \overline{X}^\vee$ over H_∞.

The i-th LO gap $\mu_i(X)$ are the coefficients of the Chow class of $F \subset B \cong \mathbb{P}^n$:

$$[F] = \sum_{i=0}^{d} \mu_i(X)[\mathbb{P}^i] \in A_*(B).$$

The LO threshold & LO gap

Theorem 2 (G. '26+)

Let $d_F = \dim(F)$.

1. $\sigma_i(X) + \mu_i(X) = \delta_i(\overline{X})$ *for all* $i = 0, \ldots, d$.
2. $\mu_i(X) = 0$ *for all* $i > d_F$, *and* $\mu_i(X) > 0$ *for all* $i \leq d_F$.
3. *If* $H_\infty \in \overline{X}^\vee$, *then* $\operatorname{codim}(\overline{X}^\vee) \leq d_F < d$.
4. $H_\infty \notin \overline{X}^\vee$ *if and only if* $\mu_i = 0$ *for all* $i = 0, \ldots, d$.

This generalizes results of Maxim-Rodriguez-Wang-Wu.[4]

The LO threshold of X is $\operatorname{LOthresh}(X) = d_F + 1$.

[4] L G Maxim, J I Rodriguez, B Wang, and L Wu. Linear optimization on varieties and Chern-Mather classes, Advances in Mathematics 437 (2024) Paper No. 109443, 22.

Example: (2_3)

Let $X \subset \mathbb{C}^4$ be the variety with ideal

$$I(X) = \left\langle 3x_1x_3 - x_2^2, 9x_1x_4 - x_2x_3, 3x_2x_4 - x_3^2, x_1 + x_2 + x_3 + x_4 - 1 \right\rangle .$$

Table: The sectional LO degrees σ_i, LO gaps μ_i, and polar degrees δ_i of X.

i	$\sigma_i(X)$	$\mu_i(X)$	$\delta_i(\overline{X})$
0	2	2	4
1	3	0	3

$\deg(X) = \sigma_d = \delta_d = 3$, $\operatorname{codim}(\overline{X}^{\vee}) = 1$, $\operatorname{LOdeg}(X) = \sigma_0 = 2$, and $\operatorname{LOthresh}(X) = 1$.

Independence Models

Let $m, r \in \mathbb{N}^k$ with each $m_i \geq 2$. Let $(m_i)_{r_i}$ denote a multinomial distribution with m_i possible outcomes and r_i trials. Consider the joint probability distributions of k independent multinomial distributions $(m_1)_{r_1}, \ldots, (m_k)_{r_k}$ as points in $\mathbb{C}^n$, where $n = \prod_{j=1}^{k} \binom{m_j + r_j - 1}{r_j}$.

The independence model $\mathcal{M}_{m,r} \subset \mathbb{C}^n$ is the Zariski closure of these distributions. It has dimension $d = |m| - k$.

The Segre-Veronese variety $\mathcal{V}_{m,r} \subset \mathbb{P}^{n-1}$ is the image of the embedding

$$\mathbb{P}^{m_1 - 1} \times \cdots \times \mathbb{P}^{m_k - 1} \hookrightarrow \mathbb{P}^{n-1}$$

given by the complete linear system $|\mathcal{O}(r_1, \ldots, r_k)|$.

Lemma 3

$\delta_i(\overline{\mathcal{M}_{m,r}}) = \delta_i(\mathcal{V}_{m,r})$ for all $i = 0, \ldots, d$. When defined, $\Delta_{\overline{\mathcal{M}_{m,r}}}$ and $\Delta_{\mathcal{V}_{m,r}}$ differ by a specific linear change of variables.

Example: $(2, 2)$

Let $k = 2$, $m = (2, 2)$, $r = (1, 1)$, so $d = 2$ and $n = 4$. The joint probability distributions are

$$(pq, p(1-q), (1-p)q, (1-p)(1-q)) \in \mathbb{C}^4 \quad \forall\, p, q \in [0, 1].$$

The ideal of $\mathcal{M}_{m,r} \subset \mathbb{C}^n$ is generated by

$$x_1x_4 - x_2x_3, x_1 + x_2 + x_3 + x_4 - 1.$$

The ideal of $\mathcal{V}_{m,r} \subset \mathbb{P}^{n-1}$ is generated by

$$x_1x_4 - x_2x_3.$$

$\mathcal{V}_{m,r}$ is the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$.

Example: (2_3)

Let $k = 1$, $m = (2)$, $r = (3)$, so $d = 1$ and $n = 4$. The joint probability distributions are

$$(p^3, 3p^2(1-p), 3p(1-p)^2, (1-p)^3) \in \mathbb{C}^4 \quad \forall\ p \in [0,1].$$

The ideal of $\mathcal{M}_{m,r} \subset \mathbb{C}^n$ is generated by

$$3x_1x_3 - x_2^2, 9x_1x_4 - x_2x_3, 3x_2x_4 - x_3^2, x_1 + x_2 + x_3 + x_4 - 1.$$

The ideal of $\mathcal{V}_{m,r} \subset \mathbb{P}^{n-1}$ is generated by

$$x_1x_3 - x_2^2, x_1x_4 - x_2x_3, x_2x_4 - x_3^2.$$

$\mathcal{V}_{m,r}$ is the twisted cubic, the degree 3 Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$.

Polar & Sectional LO Degrees of Independence Models

For the independence model $\mathcal{M}_{m,r}$, write $l = (m_1 - 1, \dots, m_k - 1)$.
Let $i = 0, \dots, |l| = \sum_{i=1}^{k} l_i = \dim(\mathcal{M}_{m,r})$.

Theorem 4 (Sodomaco '20[5])

The i-th polar degree of the Segre-Veronese variety $\mathcal{V}_{m,r}$ is

$$\delta_i(\mathcal{V}_{m,r}) = \sum_{j=0}^{|l|-i} (-1)^j \binom{|l|+1-j}{i+1} \sum_{\substack{\gamma=(\gamma_s)\\ 0\le\gamma_s\le l_s\\ |\gamma|=j}} \binom{|l|-j}{l_1-\gamma_1, \dots, l_k-\gamma_k} \prod_{s=1}^{k} \binom{l_s+1}{\gamma_s} r_s^{l_s-\gamma_s}.$$

Theorem 5 (G. '26+)

The i-th sectional LO degree of the independence model $\mathcal{M}_{m,r}$ is

$$\sigma_i(\mathcal{M}_{m,r}) = \sum_{j=0}^{|l|-i} (-1)^j \binom{|l|-j}{i} \sum_{\substack{\gamma=(\gamma_s)\\ 0\le\gamma_s\le l_s\\ |\gamma|=j}} \binom{|l|-j}{l_1-\gamma_1, \dots, l_k-\gamma_k} \prod_{s=1}^{k} \binom{l_s}{\gamma_s} r_s^{l_s-\gamma_s}.$$

[5] Published in: K Kozhasov, A Muniz, Y Qi, and L Sodomaco. On the minimal algebraic complexity of the rank-one approximation problem for general inner products, Mathematics of Computation (2025).

Example: $(2,2)$

For $m = (2,2)$, $r = (1,1)$, $\mathcal{M}_{m,r} \subset \mathbb{C}^n$ for $n = 4$ has dimension $d = 2$.

Table: The sectional LO degrees σ_i, LO gaps μ_i, and polar degrees δ_i of the independence model $(2,2)$.

i	$\sigma_i(\mathcal{M}_{m,r})$	$\mu_i(\mathcal{M}_{m,r})$	$\delta_i(\overline{\mathcal{M}_{m,r}})$
0	1	1	2
1	2	0	2
2	2	0	2

$\deg(\mathcal{M}_{m,r}) = \sigma_d = \delta_d = 2$, $\mathrm{codim}(\overline{\mathcal{M}_{m,r}}^{\vee}) = 1$,
$\mathrm{LOdeg}(\mathcal{M}_{m,r}) = \sigma_0 = 1$, and $\mathrm{LOthresh}(\mathcal{M}_{m,r}) = 1$.

Example: $(2,2)$

$\Delta_{\overline{\mathcal{M}_{m,r}}}$ is a polynomial in $a_0, \ldots, a_4$ with 6 terms of total degree 2 and its degree in a_0 is 1:

$$(a_1 - a_2 - a_3 + a_4) \cdot a_0 + a_1 a_4 - a_2 a_3$$

The LO degree of $\mathcal{M}_{m,r}$ is $\sigma_0(\mathcal{M}_{m,r}) = 1$.

$\mu_0(\mathcal{M}_{m,r}) = 1$. $\delta_0(\overline{\mathcal{M}_{m,r}}) = 2$.

Example: (2_3)

For $m = (2)$, $r = (3)$, $\mathcal{M}_{m,r} \subset \mathbb{C}^n$ for $n = 4$ has dimension $d = 1$.

Table: The sectional LO degrees σ_i, LO gaps μ_i, and polar degrees δ_i of the independence model (2_3).

i	$\sigma_i(\mathcal{M}_{m,r})$	$\mu_i(\mathcal{M}_{m,r})$	$\delta_i(\overline{\mathcal{M}_{m,r}})$
0	2	2	4
1	3	0	3

$\deg(\mathcal{M}_{m,r}) = \sigma_d = \delta_d = 3$, $\operatorname{codim}(\overline{\mathcal{M}_{m,r}}^{\vee}) = 1$,
$\operatorname{LOdeg}(\mathcal{M}_{m,r}) = \sigma_0 = 2$, and $\operatorname{LOthresh}(\mathcal{M}_{m,r}) = 1$.

Example: (2_3)

$\Delta_{\overline{\mathcal{M}_{m,r}}}$ is a polynomial in $a_0, \ldots, a_4$ with 27 terms of total degree 4 and its degree in a_0 is 2:

$$\begin{aligned}(a_1^2 - 6a_1a_2 + 9a_2^2 + 6a_1a_3 - 18a_2a_3 \\ +9a_3^2 - 2a_1a_4 + 6a_2a_4 - 6a_3a_4 + a_4^2) \cdot a_0^2 \\ + (4a_2^3 - 6a_1a_2a_3 - 6a_2^2a_3 + 12a_1a_3^2 - 6a_2a_3^2 + 4a_3^3 + 2a_1^2a_4 \\ - 6a_1a_2a_4 + 12a_2^2a_4 - 6a_1a_3a_4 - 6a_2a_3a_4 + 2a_1a_4^2) \cdot a_0 \\ - 3a_2^2a_3^2 + 4a_1a_3^3 + 4a_2^3a_4 - 6a_1a_2a_3a_4 + a_1^2a_4^2\end{aligned}$$

The LO degree of $\mathcal{M}_{m,r}$ is $\sigma_0(\mathcal{M}_{m,r}) = 2$.

$\mu_0(\mathcal{M}_{m,r}) = 2$. $\delta_0(\overline{\mathcal{M}_{m,r}}) = 4$.

The $2 \times \cdots \times 2$ Hyperdeterminant & Derangements

For $m = (2, \ldots, 2), r = (1, \ldots, 1) \in \mathbb{Z}^k$, $\mathcal{M}_{m,r}$ is the k-bit independence model. The discriminant $\Delta_k = \Delta_{\mathcal{V}_{m,r}}$ is the *hyperdeterminant of format* $2 \times \cdots \times 2$.[6] The total degree of Δ_k is

$$\delta_0(\mathcal{V}_{m,r}) = \sum_{j=0}^{k} (-1)^j (k+1-j) \binom{k}{j} (k-j)! 2^j.$$

$0, 2, 4, 24, 128, 880, 6816, 60032$ for $k = 1, \ldots, 8$.

[6] See Chapter 14: I M Gel'fand, M M Kapranov, and A V Zelevinsky. Discriminants, resultants, and multidimensional determinants, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1994.

Corollary 6

The degree in the variable a_0 of Δ_k is

$$\mathrm{LOdeg}(\mathcal{M}_{m,r}) = \sigma_0(\mathcal{M}_{m,r}) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)!.$$

$0, 1, 2, 9, 44, 265, 1854$ for $k = 1, \ldots, 8$. $\mathrm{LOdeg}(\mathcal{M})$ is the number of derangements of size k.

Thank you!

Theorem 7

Let $A \in \mathbb{Z}^{d\times n}$ be a $d \times n$ nonnegative integer matrix of rank d. Write $P = \mathrm{conv}(A)$ for the convex hull of A. Suppose that the following hold.

1. *The first column of A is zero, i.e., $a_1 = 0 \in \mathbb{Z}^d$.*
2. *A contains all lattice points of P, that is, $P \cap \mathbb{Z}^d = \{a_1, \ldots, a_n\}$.*
3. *The projective toric variety X_A is smooth.*

Then $[\mathbb{Z}^d : \mathbb{Z}A] \cdot \sigma_i(Y_A)$ is equal to the mixed volume

$$MV(\underbrace{P, \cdots, P}_{i}, \mathrm{Newt}(\partial_1\varphi_A) \times \Delta_i, \ldots, \mathrm{Newt}(\partial_d\varphi_A) \times \Delta_i), \quad (1)$$

where $\Delta_i = \mathrm{conv}\{0, e_1, \ldots, e_i\} \subset \mathbb{R}^i$ is the standard simplex in $\mathbb{R}^i$. Furthermore, all critical points are smooth and lie in the algebraic torus $(\mathbb{C}^*)^n$ of Y_A.

Connection to Gröbner Degeneration

Let $I = I(N_{\overline{X}})$. If $f \in I$, $\mathrm{Lt}_w(I)$ is sum of terms of f where a_0 has degree $\deg_{a_0}(f)$. Set $\mathrm{Lt}_w(I) = \langle \mathrm{Lt}_f(I) \mid f \in I \rangle$.

$Y = V(\mathrm{Lt}_w(I))$, $Y_\infty = Y \cap B$, $Y_{\mathrm{aff}} = \overline{Y \setminus Y_\infty}$.

Corollary 8 (G. '26+)

1. $[Y] = [Y_\infty] + [Y_{\mathrm{aff}}]$.
2. $[Y] = [N_{\overline{X}}] = \sum_{i=0}^{d} \delta_i(X)[\mathbb{P}^i \times \mathbb{P}^{n-1-i}] \in A_*(\mathbb{P}^n \times (\mathbb{P}^n)^\vee)$.
3. $[Y_\infty] = [F] = \sum_{i=0}^{d} \mu_i(X)[\mathbb{P}^i] \in A_*(B)$.
4. $[Y_{\mathrm{aff}}] = \sum_{i=0}^{d} \sigma_i(X)[\mathbb{P}^i \times \mathbb{P}^{n-i}] \in A_*(\mathbb{P}^n \times (\mathbb{P}^n)^\vee)$.

Counting Nash Equilibria

Corollary 9

Let $m = (m_1, \ldots, m_k)$*,* $r = (1, \ldots, 1) \in \mathbb{Z}^k$*. The maximum number of totally mixed Nash equilibria in a* k*-player game where player* i *has exactly* m_i *pure strategies is* $\mathrm{LOdeg}(\mathcal{M}_{m,r})$*.*